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Lax form of the q-Painlevé equation associated with the $A_2^{(1)}$ surface

Hidetaka Sakai

Graduate School of Mathematical Sciences, University of Tokyo, Komaba, Tokyo 153-8914, Japan

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Abstract
In this paper, the q-difference Painlevé equation whose space of initial conditions is the $A_2^{(1)}$ surface is shown to appear as a particular case of a q-Garnier system, which is a q-difference counter part of monodromy preserving deformation in the generic situation. As a consequence, the q-Painlevé equation of $A_2^{(1)}$ is written in a Lax formalism.

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1. Introduction

There are many discrete analogues of the Painlevé differential equations. Many of them were discovered by Ramani and Grammaticos and their co-workers, as equations which pass through the singularity confinement test [7]. The singularity confinement test is regarded as a discrete counterpart of the Painlevé test.

A classification of the discrete Painlevé equations with a view of the theory of rational surfaces is also known [9]. While I prefer to call equations by the types of surfaces because of the uniqueness of their correspondence, there are many researchers who call them by their symmetries. Hence we write both of the lists.

Surface	$A_0^{(1)}$	$A_0^{(1)*}$	$A_1^{(1)}$	$A_2^{(1)}$	$A_3^{(1)}$	$A_4^{(1)}$	$A_5^{(1)}$	$A_6^{(1)}$	$A_7^{(1)}$	$A_7^{(1)\prime}$	$A_8^{(1)}$
Symmetry	$E_8^{(1)}$	$E_8^{(1)}$	$E_7^{(1)}$	$E_6^{(1)}$	$D_5^{(1)}$	$A_4^{(1)}$	$(A_2+A_1)^{(1)}$	$(A_1+A_1)^{(1)}$	$A_1^{(1)}$	$A_1^{(1)}$	–
$A_0^{(1)**}$	$A_1^{(1)*}$	$A_2^{(1)*}$	$D_4^{(1)}$	$D_5^{(1)}$	$D_6^{(1)}$	$D_7^{(1)}$	$D_8^{(1)}$	$E_6^{(1)}$	$E_7^{(1)}$	$E_8^{(1)}$	
$E_8^{(1)}$	$E_7^{(1)}$	$E_6^{(1)}$	$D_4^{(1)}$	$A_3^{(1)}$	$(A_1+A_1)^{(1)}$	$A_1^{(1)}$	–	$A_2^{(1)}$	$A_1^{(1)}$	–	

However, Lax forms of some of discrete Painlevé equations have not been obtained yet. The aim of this paper is to present a Lax pair of q-Painlevé equation associated with the $A_2^{(1)}$ surface.

Theory of the Painlevé differential equations has developed through two important aspects. One is the classification of second-order algebraic ordinary differential equations of normal type that satisfy the Painlevé property. The other one is a deformation theory of linear ordinary differential equations. Painlevé and Gambier completed the first one and obtained the six Painlevé differential equations. On the other hand, Fuchs reached the sixth Painlevé equation from a completely different problem, deformation theory of linear equations. Going into detail, we see that the sixth equation appears as the condition that we move the coefficients of the second-order Fuchsian equation having four regular singularities without changing its monodromy [2].

This result of Fuchs was generalized afterwards by Garnier and Schlesinger. A result of Garnier is connected to the deformation theory of the second-order linear equation with irregular singularities. He obtained the other five Painlevé equations from this consideration [3]. Schlesinger considered the isomonodromic deformation of an $m \times m$-linear system of first-order differential equations with regular singularities [11]. At a later time Jimbo, Miwa and Ueno established a general theory of monodromy preserving deformation for the matrix system of first-order differential equations with regular and irregular singularities [4, 5]. In their theory the Painlevé equations are written in the form of a compatibility condition between a 2×2-linear system and an associated deformation system. We call this description 'Lax form' of the Painlevé equations.

We see some merits that we could express the Painlevé equations in their Lax form. First of all, linear differential equations are easy to be identified with their data of singularities; in particular, the classification of the Painlevé equations corresponds with coalescence of singularities of linear differential equations. Besides, particular solutions of Riccati type appear where the monodromy of linear equations is reducible; we obtain a key for particular solutions from studies of associated linear equations.

Getting back to the discrete case, we consider these two important aspects. Singularity confinement, which was presented by Ramani and Grammaticos *et al* is a discretization of the Painlevé property. Then, how about the other one, Lax form?

There are three types of discrete Painlevé equations: elliptic-difference, q-difference and difference. As concerns difference Painlevé equations, in particular $D_l^{(1)}$ and $E_l^{(1)}$ types, they possess the same rational surfaces that the Painlevé differential equations have as their spaces of initial conditions. Difference equations can be regarded as contiguity relations, i.e. Bäcklund transformations of the Painlevé differential equations. We can lift up these relations to associated linear equations; we see them as discrete deformation (Schlesinger transformation) of linear differential equations and also as coming from compatibilities of two discrete deformations of linear differential equations.

Although the difference equations of types $A_0^{(1)**}$, $A_1^{(1)*}$ and $A_2^{(1)*}$ do not correspond to any Painlevé differential equation, the author believes that they should correspond to the Garnier system or degenerated Garnier systems; they should be written in the framework of Schlesinger transformations, which is generally studied in Jimbo and Miwa's paper [5]. Recently, Arinkin and Borodin calculated a Lax pair of difference Painlevé equation of $A_2^{(1)*}$ type, and in fact, they show that the system can be regarded as a discrete deformation of a linear differential equation, though they did not give explicit form of this linear differential equation [1].

Therefore, if we want a new deformation equation different from Schlesinger transformations which appear in the paper of Jimbo and Miwa's, the author thinks, that would be elliptic-difference or q-difference. In the paper of Jimbo and the author, they studied q-analogue of Fuchs' result, that is, a deformation theory of linear q-difference equation [6]. Recently, q-analogue of the Garnier system, which is a higher dimensional extension of Fuchs' result, was also studied [10]. For our information, the Garnier system is equivalent to the Schelesinger system of rank 2.

Just like the differential case, they are quite natural and general situation; so the q-Painlevé equation should be included among the q-Garnier system or its degenerations if they could be written in a 2×2-Lax form. However, while the most generic Painlevé equation, the sixth, coincides with the Garnier system of two dimensional ($N = 1$), the two-dimensional q-Garnier system coincides with the q-Painlevé equation of $A_3^{(1)}$ type; more generic equations, $A_0^{(1)*}$, $A_1^{(1)}$ and $A_2^{(1)}$, do not appear.

In this paper, we see that the q-Painlevé equation of $A_2^{(1)}$ type appear as a particular case of the four-dimensional q-Garnier system ($N = 2$). This construction owes much to calculations in Arinkin and Borodin's paper [1]. The same problem for $A_0^{(1)*}$ and $A_1^{(1)}$ still remains open.

The text is organized as follows. In the following section we show that the q-Painlevé equation of $A_2^{(1)}$ type appear as a particular case of the four-dimensional q-Garnier system; we calculate the Lax pair explicitly in the final section.

2. The q-Painlevé equation as a particular case of the q-Garnier system with $N = 2$

The Garnier system is a multi-variable extension of the sixth Painlevé differential equation; we can identify it with the Schlesinger system with rank 2. We studied a q-analogue of the Garnier system in the previous paper [10]. It arises as the condition for preserving the connection matrix of linear q-difference equations. In this section, we would show that the q-Painlevé equation of type $A_2^{(1)}$ appear as a particular case for the q-Garnier system of four dimension ($N = 2$).

In the case that $N = 2$, we go further in detail. We can express the q-Garnier system as the dynamical system

$$\left(y_1, y_2, y_3, z_1, z_2, z_3; \begin{matrix} a_1, a_2, a_3, a_4, a_5, a_6 \\ \kappa_1, \kappa_2, \theta_1, \theta_2 \end{matrix}\right) \mapsto \left(\bar{y}_1, \bar{y}_2, \bar{y}_3, \bar{z}_1, \bar{z}_2, \bar{z}_3; \begin{matrix} a_1, a_2, a_3, a_4, qa_5, qa_6 \\ \kappa_1, \kappa_2, q\theta_1, q\theta_2 \end{matrix}\right),$$

where the parameters satisfy the following relation:

$$\kappa_1\kappa_2 \prod_{i=1}^{6} a_i = \theta_1\theta_2.$$

Now $\bar{z}_n$ and $\bar{y}_n$ are determined by equations

$$\frac{\overline{z_n}}{z_5 z_6}\left(\frac{z_n - z_5}{a_n - a_5} - \frac{z_n - z_6}{a_n - a_6}\right) = \left(1 - \frac{(1 - q\kappa_1/\kappa_2)(a_5 - a_6)}{z_5 - z_6}\right)\left(\frac{z_n - z_5}{a_n - a_5}\frac{1}{z_5} - \frac{z_n - z_6}{a_n - a_6}\frac{1}{z_6}\right), \tag{2.1}$$

$$\begin{aligned}\overline{y_n}\left(1 - \frac{(1 - q\kappa_1/\kappa_2)(a_5 - a_6)}{z_5 - z_6}\right) &= -y_n \frac{(a_n - qa_5)(a_n - qa_6)}{(z_5 - z_6)^2}\left(\frac{z_n - z_5}{a_n - a_5} - \frac{z_n - z_6}{a_n - a_6}\right) \\ &\quad \times \left(\frac{w_n + z_5}{a_n - qa_5} - \frac{w_n + z_6}{a_n - qa_6}\right), \qquad (n = 1, 2, 3).\end{aligned} \tag{2.2}$$

Here

$$z_j = \frac{\kappa_2 + \sum_{n=1}^{3} \frac{y_n z_n}{\Theta'(a_n)(a_j - a_n)}}{\sum_{n=1}^{3} \frac{y_n}{\Theta'(a_n)(a_j - a_n)}}, \qquad w_j = \frac{\kappa_2}{\kappa_1}\frac{\kappa_1 + \sum_{n=1}^{3} \frac{y_n w_n}{\Theta'(a_n)(a_j - a_n)}}{\sum_{n=1}^{3} \frac{y_n}{\Theta'(a_n)(a_j - a_n)}}, \qquad (j = 4, 5, 6), \tag{2.3}$$

where $\Theta(x) = (x - a_1)(x - a_2)(x - a_3)$ and $w_n (n = 1, 2, 3)$ is the solution of equation

$$\begin{pmatrix} \frac{z_4-z_1}{a_4-a_1} & \frac{z_4-z_2}{a_4-a_2} & \frac{z_4-z_3}{a_4-a_3} \\ \frac{z_5-z_1}{a_5-a_1} & \frac{z_5-z_2}{a_5-a_2} & \frac{z_5-z_3}{a_5-a_3} \\ \frac{z_6-z_1}{a_6-a_1} & \frac{z_6-z_2}{a_6-a_2} & \frac{z_6-z_3}{a_6-a_3} \end{pmatrix} \begin{pmatrix} \frac{y_1 w_1}{(a_1-a_2)(a_1-a_3)} \\ \frac{y_2 w_2}{(a_2-a_3)(a_2-a_1)} \\ \frac{y_3 w_3}{(a_3-a_1)(a_3-a_2)} \end{pmatrix} = \begin{pmatrix} z_4 \\ z_5 \\ z_6 \end{pmatrix}. \tag{2.4}$$

Although it seems to be six dimensional, it is essentially a four-dimensional dynamical system because we have two integrals

$$\frac{y_1}{(a_1 - a_2)(a_1 - a_3)} + \frac{y_2}{(a_2 - a_3)(a_2 - a_1)} + \frac{y_3}{(a_3 - a_1)(a_3 - a_2)} = \kappa_2, \tag{2.5}$$

$$\frac{(w_1 + z_1)y_1}{(a_1 - a_2)(a_1 - a_3)a_1} + \frac{(w_2 + z_2)y_2}{(a_2 - a_3)(a_2 - a_1)a_2} + \frac{(w_3 + z_3)y_3}{(a_3 - a_1)(a_3 - a_2)a_3} = \kappa_1 + \kappa_2 + \frac{\theta_1 + \theta_2}{a_1 a_2 a_3}. \tag{2.6}$$

Remark 2.1. The q-Garnier system can be equivalently rewritten as

$$\overline{z_n}\left(\frac{z_n - z_5}{a_n - a_5} - \frac{z_n - z_6}{a_n - a_6}\right) + \left(\frac{z_n - z_5}{a_n - a_5}\overline{w_5} - \frac{z_n - z_6}{a_n - a_6}\overline{w_6}\right) = 0, \tag{2.7}$$

$$\frac{\bar{y}_n \bar{w}_5}{y_n z_6} = \frac{(a_n - qa_5)(a_n - qa_6)}{(z_5 - z_6)^2}\left(\frac{z_n - z_5}{a_n - a_5} - \frac{z_n - z_6}{a_n - a_6}\right)\left(\frac{w_n + z_5}{a_n - qa_5} - \frac{w_n + z_6}{a_n - qa_6}\right), \quad (n = 1, 2, 3). \tag{2.8}$$

Inversely the q-Garnier system is derived from equations (2.7)–(2.8) and equation (2.5). Here we used equations

$$\frac{\overline{w_5}}{z_6} = \frac{\overline{w_6}}{z_5} = \frac{(1 - q\kappa_1/\kappa_2)(a_5 - a_6)}{z_5 - z_6} - 1,$$

which is derived from condition (2.5) and (2.7)–(2.8).

In order to obtain the q-Painlevé equation associated with the $A_2^{(1)}$ surface, we apply, to equations (2.7)–(2.8), the following condition instead of equation (2.5):

$$\frac{y_i}{a_i - a_j} + \frac{y_j}{a_j - a_i} = \kappa_2, \qquad i \neq j, \quad i, j \in \{1, 2, 3\}. \tag{2.9}$$

The conditions imply that y_n can be parameterized by one variable λ, that is, we can put

$$y_n = \kappa_2(a_n - \lambda), \qquad n = 1, 2, 3. \tag{2.10}$$

The other variables, $z_n (n = 1, 2, 3)$, with condition (2.6), can be parameterized by two variables. When we put

$$\mu = -\left(1 + \sum_{n=1}^{3} \frac{z_n}{\Theta'(a_n)}\right)\frac{y_1 y_2 y_3}{\kappa_2^3}, \tag{2.11}$$

$$\gamma = \left(\sum_{n=1}^{3} \frac{z_n}{\Theta'(a_n)}\right)\lambda - a_1 - a_2 - a_3 + \sum_{n=1}^{3} \frac{a_n z_n}{\Theta'(a_n)}, \tag{2.12}$$

then z_n is expressed as follows:

$$z_n = a_n^2 + (\gamma + \lambda)a_n + \delta + \frac{\mu}{a_n - \lambda}, \tag{2.13}$$

where

$$\delta = \frac{1}{\kappa_1-\kappa_2}\left[\kappa_1(2\lambda^2-\sigma_1\lambda+\sigma_2+\gamma(\gamma+\sigma_1)) - \frac{1}{\lambda}(\kappa_1\tilde{\mu}+\kappa_2\mu-\theta_1-\theta_2)\right],$$
$$\tilde{\mu} = \frac{1}{\mu}\prod_{i=1}^{6}(\lambda-a_i), \qquad \sigma_1 = \sum_{i=1}^{6} a_i, \qquad \sigma_2 = \sum_{i<j} a_i a_j. \tag{2.14}$$

Conditions (2.9) are invariant under the time evolution of the q-Garnier system when the following condition is satisfied:

$$q\kappa_1 = \kappa_2. \tag{2.15}$$

Theorem 2.1. *The dynamical system, (2.7)–(2.8), with conditions, (2.6), (2.9) and (2.15), can be rewritten by the following form:*

$$\left(\lambda,\nu,\gamma;\begin{matrix}a_1,a_2,a_3,a_4,a_5,a_6\\ \kappa_2,\theta_1,\theta_2\end{matrix}\right) \mapsto \left(\bar{\lambda},\bar{\nu},\bar{\gamma};\begin{matrix}a_1,a_2,a_3,a_4,qa_5,qa_6\\ \kappa_2,q\theta_1,q\theta_2\end{matrix}\right),$$
$$(\lambda-\underline{\nu})(\lambda-\nu) = \frac{(\lambda-a_1)(\lambda-a_2)(\lambda-a_3)(\lambda-a_4)}{(\lambda-a_5)(\lambda-a_6)}, \tag{2.16}$$

$$\left(1-\frac{\nu}{\bar{\lambda}}\right)\left(1-\frac{\nu}{\lambda}\right) = \frac{a_5a_6}{q}\frac{(\nu-a_1)(\nu-a_2)(\nu-a_3)(\nu-a_4)}{\left(a_5a_6\nu+\frac{\theta_1}{\kappa_2}\right)\left(a_5a_6\nu+\frac{\theta_2}{\kappa_2}\right)}, \tag{2.17}$$

$$a_5a_6\lambda\bar{\lambda}(a_1+a_2+a_3+a_4+\bar{\gamma}-\nu)(a_5+a_6+\gamma+\nu)+q(a_5a_6\nu+\theta_1/\kappa_2)(a_5a_6\nu+\theta_2/\kappa_2) = 0. \tag{2.18}$$

Remark 2.2. The q-Painlevé equation of type $A_2^{(1)}$ can be expressed as a dynamical system $(\lambda,\nu)\mapsto(\bar{\lambda},\bar{\nu})$, where $\bar{\lambda}$ and $\bar{\nu}$ are determined by equations (2.16)–(2.17). Note that the two equations do not contain the variable γ.

This q-difference system were found by Grammaticos and Ramani and their co-workers as in the case of many other discrete Painlevé equations (cf [8]). They call it the asymmetric q-P_V equation.

It may be difficult to grasp the meaning of calculus described in this section. Because the q-Garnier system is derived from the Lax pair, it is getting easier to understand by seeing the corresponding conditions on the linear equations with respect to each of these conditions. In the following section we introduce a view of deformation theory of linear q-difference equations.

3. The Lax pair

Consider a 2×2 matrix system with polynomial coefficients

$$Y(qx) = A(x)Y(x). \tag{3.1}$$

A deformation theory of a linear q-difference equation in generic situation was studied in the previous papers [6, 10]. In the theory of the monodromy preserving deformation of Fuchsian equations, an extra parameter $t=(t_j)$ is introduced denoting the position of regular singular points. In the formulation, in terms of q-difference equations, we put the (discrete) deformation parameters at zeros of $\det A(x)$, the eigenvalues of the leading term, and the eigenvalues of the constant term.

The connection preserving deformation of the linear q-difference equation, which is a discrete counterpart of monodromy preserving deformation, is equivalent to the existence of a linear deformation equation whose coefficients are rational in x. We express the deformation equation in the form

$$\overline{Y(x)} = B(x)Y(x), \tag{3.2}$$

and can express the q-Schlesinger equation in the form

$$\overline{A(x)}B(x) = B(qx)A(x) \tag{3.3}$$

by the compatibility of the deformation equation and the original linear q-difference equation. Here the bar represents a discrete time evolution.

We now take $A(x)$ to be of the form

$$A(x) = A_0 + A_1x + A_2x^2 + A_3x^3, \tag{3.4}$$

$$A_3 = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}, \qquad A_0(t) \text{ has eigenvalues } \theta_1, \theta_2, \tag{3.5}$$

$$\det A(x) = \kappa_1\kappa_2 \prod_{i=1}^{6}(x - a_i). \tag{3.6}$$

Clearly we have

$$\kappa_1\kappa_2 \prod_{i=1}^{6} a_i = \theta_1\theta_2. \tag{3.7}$$

Define y_i, z_i and w_i $(i = 1, 2, \ldots, 6)$ by

$$A(a_i) = y_i \begin{pmatrix} 1 \\ w^{-1}z_i \end{pmatrix} (w_i \quad w) \qquad (i = 1, 2, \ldots, 6). \tag{3.8}$$

(Note that $\det A(a_i) = 0$.) Then there are 18 parameters and a variable of gauge freedom, w. They are redundant. The variables, z_4, z_5, z_6 and w_i $(i = 1, 2, \ldots, 6)$, are determined by relations (2.3) and (2.4). Moreover, equation (2.6) is also satisfied.

Condition (2.9) is equivalent to the condition that the $(1, 2)$ element of the matrix, $A(x)$, is a polynomial of degree 1 in x. That is, when we write the coefficient matrix as

$$A(x) = \begin{pmatrix} \kappa_1 W(x) & \kappa_2 wL(x) \\ \kappa_1 w^{-1}X(x) & \kappa_2 Z(x) \end{pmatrix},$$

then $L(x)$ is written by variable, λ, as $L(x) = x - \lambda$.

Besides, when we put $Z(x) = L(x)\bigl(x^2 + (\gamma + \lambda)x + \delta + \frac{\mu}{L(x)}\bigr)$, variables, μ, γ and δ, are determined by equations (2.11), (2.12) and (2.14).

Here we consider a deformation of this linear q-difference system. We look at the condition that $L(x)$ remains of degree 1 under the deformation of $A(x)$, which is determined by equation (3.3).

Proposition 3.1. *The* $(1, 2)$ *element,* $L(x)$*, remains of degree 1 under the deformation, if* $q\kappa_1 = \kappa_2$.

Theorem 3.2. *The q-Schlesinger equation, (3.3), satisfying condition of parameters,* $q\kappa_1 = \kappa_2$*, has a particular solution which is written by the form of (2.16)–(2.18) with the gauge satisfying*

$$\frac{\bar{w}}{w} = \frac{\bar{w}_5 - \bar{w}_6}{z_5 - z_6}.$$

In particular, the matrix B is written by the terms of elements of A as follows:

$$B(x) = \frac{x(x1 + B^\circ)}{(x - qa_5)(x - qa_6)}, \qquad B(x)^{-1} = 1 + \frac{\widetilde{B^\circ}}{x} \tag{3.9}$$

$$B^\circ = q\begin{pmatrix} -\frac{a_6z_5 - a_5z_6}{z_5 - z_6} & w\frac{a_5 - a_6}{z_5 - z_6} \\ w^{-1}\frac{a_5 - a_6}{\bar{w}_5 - \bar{w}_6}\bar{w}_5 z_5 & -\frac{a_6\bar{w}_5 - a_5\bar{w}_6}{\bar{w}_5 - \bar{w}_6} \end{pmatrix}, \qquad \widetilde{B^\circ} = (\det B^\circ)(B^\circ)^{-1}. \tag{3.10}$$

Proof. The calculation is hard but straightforward. It is essentially same to the generic q-Garnier system (see [10]). We will see only the proof of the proposition.

From the q-Schlesinger system (3.3), the coefficient matrix is determined as

$$\overline{A(x)} = B(qx)A(x)B(x)^{-1}.$$

Continuing the calculus, we obtain

$$\begin{aligned}\overline{A(x)} &= \frac{1}{q(x - a_5)(x - a_6)}(qx1 + B^\circ)A(x)(x1 + \widetilde{B^\circ}) \\ &= \frac{1}{q(x - a_5)(x - a_6)}(qA_3x^5 + (qA_2 + B^\circ A_3 + qA_3\widetilde{B^\circ})x^4 + \cdots).\end{aligned}$$

On the other hand, we know that $\overline{A(x)}$ is of the form (see [10])

$$\begin{aligned}\overline{A(x)} &= \overline{A_3}x^3 + \overline{A_2}x^2 + \overline{A_1}x + \overline{A_0} \\ &= \frac{1}{q(x - a_5)(x - a_6)}(q\overline{A_3}x^5 + q(\overline{A_2} - (a_5 + a_6)\overline{A_3})x^4 + \cdots).\end{aligned}$$

Comparing these, we have $\overline{A_2} = A_2 + (a_5 + a_6)\overline{A_3} + q^{-1}B^\circ A_3 + A_3\widetilde{B^\circ}$. Since $(1, 2)$ elements of A_2 and $\overline{A_3} = A_3$ are zero, vanishing of $(1, 2)$ element of $q^{-1}B^\circ A_3 + A_3\widetilde{B^\circ}$ is sufficient. The $(1, 2)$ element of $q^{-1}B^\circ A_3 + A_3\widetilde{B^\circ}$ is

$$(\kappa_2 - q\kappa_1)w\frac{a_5 - a_6}{z_5 - z_6}.$$

Therefore, if $q\kappa_1 = \kappa_2$, then $L(x)$ remains of degree 1 in x. □

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